SPARKCHARTS

CALCULUS II

OVERVIEW

Calculus is the study of "nice"—smoothly changing—functions.

- **Differential calculus** studies how quickly a function is changing at a particular point. For more on differential calculus, see the *Calculus* I SparkChart.
- **Integral calculus** studies areas enclosed by curves and is used to compute a continuous (as opposed to discrete) summation. Integration is used in geometry to find the length of an arc, the area of a surface, and the volume of a solid; in physics, to compute the total work done by a varying force or the location of the center of mass of an irregular object; in statistics, to work with varying probabilities.
- **Differential equations** (diff-eqs) express a relationship between a function and its derivatives. Diff-eqs come up when modeling natural phenomena.
- **Infinite series** are special types of functions. Familiar functions can often be represented as infinite **Taylor polynomials.** Infinite series are used to differentiate and integrate difficult functions, as well as to approximate values of functions and their derivatives.

REVIEW OF TERMS

- A **function** is a rule that assigns to each value of the **domain** a unique value of the **range**.
- Function $f(x)$ is **continuous** on some interval if whenever x_1 is close to x_2, $f(x_1)$ is close to $f(x_2)$.
- Function $f(x)$ is **increasing** on some interval if whenever $x_1 < x_2$, $f(x_1) < f(x_2)$ (so $f'(x)$ is positive). It is **decreasing** if $f(x_1) > f(x_2)$ (so $f'(x)$ is negative). A function that either never increases or never decreases is called **monotonic**.
- Function $f(x)$ is **differentiable** on some open interval if its derivative exists everywhere on that interval. A differentiable function must be continuous, and it cannot have vertical tangents on the interval.
- Function $f(x)$ is **concave up** on some interval if its second derivative $f''(x)$ is positive there; its graph "cups up." It is **concave down** if $f''(x)$ is negative; its graph "cups down."
- The line $x = a$ is a **vertical asymptote** for $f(x)$ if $f(x)$ "blows up" to (positive or negative) infinity as x gets closer and closer to a (from the left side, the right side, or both). Formally, $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ (or both).
- The line $y = b$ is a **horizontal asymptote** for $f(x)$ if the value of $f(x)$ gets close to b as $|x|$ becomes very large (when x is positive, negative, or both). Formally, $\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ (or both).

AREA UNDER A CURVE AND THE DEFINITE INTEGRAL

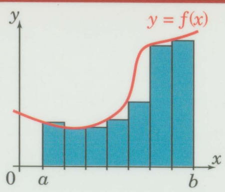
BASIC PROBLEM OF INTEGRAL CALCULUS

Given a function $y = f(x)$ on the interval $[a, b]$, what is the area enclosed by this curve, the x-axis, and the two vertical lines $x = a$ and $x = b$?

NOTE: We always speak of "signed" area: a curve above the x-axis is said to enclose positive area, while a curve below the x-axis is said to enclose negative area. The concept of "negative area" may seem ridiculous, but signed area is more versatile and simpler to keep track of.

APPROXIMATIONS TO THE AREA

- **Left-hand rectangle approximation:** We can approximate this area by a series of n rectangles. Divide the interval into n equal subintervals of width $\Delta x = \frac{b-a}{n}$ and obtain $n+1$ points on the x-axis at $x_0 = a,\ x_1 = a + \Delta x,\ \ldots,\ x_n = a + n\Delta x = b$. These are the bottom corners of n rectangles, which we'll always number 0 to $n-1$. The height of each rectangle is the value of $f(x)$ at the left x-axis corner. The k^{th} rectangle has height $f(x_k)$ and area $\Delta x f(x_k)$. The total area of the n rectangles, then, is



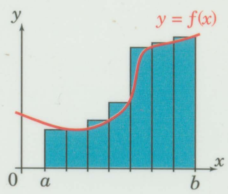
$$L_n = \Delta x\big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big) = \Delta x \sum_{k=0}^{n-1} f(x_k).$$

- Larger n will give more accurate approximation to the area.

Ex: We approximate the area under the curve $y = x^2$ on the interval $[0, 1]$ with 4 rectangles of width $\Delta x = \frac{1}{4}$ and heights $0,\ \left(\frac{1}{4}\right)^2,\ \left(\frac{1}{2}\right)^2,\ \left(\frac{3}{4}\right)^2$, for a total area of

$$L_4 = \tfrac{1}{4}\left((0)^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{4}\right)^2\right) = \tfrac{7}{32} = 0.21875.$$

- **Right-hand rectangle approximation:** Instead of taking the height of each rectangle to be the value of $f(x)$ at the left x-axis corner, we can take the the value of $f(x)$ at the right corner. The height of the k^{th} rectangle is now $f(x_{k+1})$ for a total area of



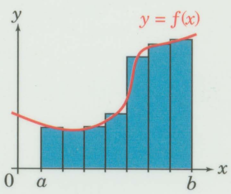
$$R_n = \Delta x\big(f(x_1) + f(x_2) + \cdots + f(x_n)\big) = \Delta x \sum_{k=1}^{n} f(x_k).$$

- Right-hand and left-hand approximations are related by $R_n = L_n + \Delta x\big(f(b) - f(a)\big)$.

Ex: For $f(x) = x^2$ on the interval $[0, 1]$,

$$R_4 = \tfrac{1}{4}\left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{4}\right)^2 + (1)^2\right) = \tfrac{15}{32} = 0.46875.$$

- **Midpoint Rule:** The height of each rectangle can be taken to be $f(x)$ evaluated at the midpoint of each rectangle; the height of the k^{th} rectangle is now

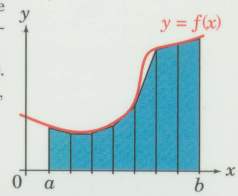


$$f\left(a + \Delta x\left(k + \tfrac{1}{2}\right)\right) = f\left(\tfrac{x_k + x_{k+1}}{2}\right)$$

for a total area of

$$M_n = \Delta x\left(f\left(\tfrac{x_0+x_1}{2}\right) + \cdots f\left(\tfrac{x_{n-1}+x_n}{2}\right)\right) = \Delta x \sum_{k=0}^{n-1} f\left(\tfrac{x_k+x_{k+1}}{2}\right).$$

- **Trapezoidal Rule:** We can approximate the area under the curve using trapezoids with the same two vertical sides and x-axis side as the rectangles. The area of a trapezoid is (*average length of two parallel sides*) $\times$ (*distance between them*). The area of the k^{th} trapezoid, then, is $\frac{\Delta x}{2}\left(f(x_k) + f(x_{k+1})\right)$, for a total area of



$$T_n = \frac{\Delta x}{2}\big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big).$$

- **Simpson's Rule:** This time, we suppose n to be even and approximate the area with $\frac{n}{2}$ parabola pieces with the k^{th} parabola defined by points on the curve at $x_{2k-2},\ x_{2k-1}$, and x_{2k}. The total area is given by

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big).$$

COMPARING APPROXIMATIONS TO THE AREA

- If $f(x)$ is increasing on the interval $[a, b]$, then $L_n <$ Area $< R_n$ for each n. If $f(x)$ is decreasing on the whole interval, the inequalities are reversed.
- If $f(x)$ is concave up on the whole interval, then L_n is a better approximation to the area than R_n. If $f(x)$ is concave down on the whole interval, then R_n is a better approximation than L_n.
- The Midpoint Rule approximates area more accurately than the Trapezoidal Rule. Both are better than the left- or right-hand rectangle approximations. Simpson's Rule is best of all.

RIEMANN SUMS

The left- and right-hand rectangle approximations and the Midpoint Rule all use a prescribed point in the subinterval as the height of the rectangle. In general, we can pick *any* **sample point** x_k^* in the k^{th} subinterval. The area approximation, then, is $\Delta x \sum_{k=0}^{n-1} f(x_k^*)$. This general area approximation is called a **Riemann sum,** and its limit as n increases will give the area of the region.

THE DEFINITE INTEGRAL

If the limit $\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(x_k^*)$ exists, then the function $f(x)$ is called **integrable** on the interval $[a, b]$. The limit represents the area under the curve and is denoted $\int_a^b f(x)\,dx$.

- In this notation, $\int$ is the **integral sign,** $f(x)$ is the **integrand,** and a and b are the **lower** and **upper limits of integration,** respectively.
- The marker dx keeps track of the variable of integration and evokes a very small Δx; intuitively, the "integral from a to b of $f(x)\,dx$" is a sum of heights (function values) times tiny widths dx (i.e., a sum of many minute areas).
- Being "integrable" says nothing about how easy the symbolic integral is to write down. Often, the integral is difficult to express.
- All functions made up of a finite number of pieces of continuous functions are integrable. In practice, every function encountered in a Calculus class will be integrable except at points where it "blows up" towards $\pm\infty$ (equivalently, has a vertical asymptote).

Properties of the definite integral: Let $f(x)$ and $g(x)$ be functions integrable on the interval $[a, b]$, and p be a point inside the interval.

1. **Sums and differences:** $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$
2. **Scalar multiples:** $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$ Here, c is any real number.
3. **Reversing the limits:** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
4. **Concatenation:** $\int_a^p f(x)\,dx + \int_p^b f(x)\,dx = \int_a^b f(x)\,dx.$
5. **Betweenness:** If $f(x) \le g(x)$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

   In particular:
   - If $f(x) \ge 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$.
   - If M is the maximum value of $f(x)$ on $[a, b]$ and m is the minimum value, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

ANTIDERIVATIVES AND THE INDEFINITE INTEGRAL

Antidifferentiation is the reverse of differentiation: an **antiderivative** of $f(x)$ is any function $F(x)$ whose derivative is equal to the original function: $F'(x) = f(x)$ in a pre-established region. Functions that differ by constants have the same derivative; therefore, we look for a **family of antiderivatives** $F(x) + C$, where C is any real constant.

The family of the antiderivatives of $f(x)$ is denoted by the **indefinite integral:**

$$\int f(x)\,dx = F(x) + C \text{ if and only if } F'(x) = f(x).$$

The indefinite integral represents a family of functions differing by constants.

"SO IF A MAN'S WIT BE WANDERING, LET HIM STUDY THE MATHEMATICS."

FRANCIS BACON

THE FUNDAMENTAL THEOREM OF CALCULUS

The **Fundamental Theorem of Calculus (FTC)** brings together differential and integral calculus.

MAIN POINT: Differentiation and integration are inverse processes. Finding antiderivatives is a lot like calculating areas under curves.

STATEMENT OF THE THEOREM

- **Part 1:** Let $f(x)$ be a function continuous on the interval $[a,b]$. Then the area function $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on $[a,b]$ and $F'(x) = f(x)$.
- **Part 2:** If $f(x)$ is a function continuous on the interval $[a,b]$ and $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$. The total change in the antiderivative function over an interval is the same as the area under the curve.

WHY IS THE FTC TRUE?

There are two ways of thinking about it:

1. The *change* in the area function (function whose value at a is the area under $f(x)$ up to a) is chronicled by $f(x)$ itself: the area function changes quickly if $f(x)$ is large, slowly if $f(x)$ is small; it is increasing whenever $f(x)$ is positive and decreasing if $f(x)$ is negative. So $f(x)$ behaves like the derivative of the function for the area under $f(x)$.

2. The function for the area under $f'(x)$ behaves like $f(x)$ itself.

- If $f(x)$ is increasing (or decreasing), then $f'(x)$ is positive (or negative) and the area under $f'(x)$ is increasing (or decreasing).
- If $f(x)$ is changing quickly, then $|f'(x)|$ is large—and the area under $f'(x)$ is changing quickly as well.
- If the growth rate of $f(x)$ is positive but slowing down to 0 (i.e., $f(x)$ is concave down approaching a local maximum), then $f'(x)$ is crossing the x-axis from the positive half-plane to the negative half-plane; at the same time, the area under $f'(x)$, which has been growing while $f'(x) > 0$, is stopping its growth and will start to decrease: like $f(x)$, the area under $f'(x)$ is nearing a local maximum.

USING THE FTC

The FTC justifies using the integral sign for both antiderivatives and areas under curves. And it gives us a simple way to calculate the area under many curves.

Ex: The area under the curve $y = x^2$ over the interval $[0,1]$ is given by $\int_0^1 x^2\,dx$. Since $F(x) = \frac{1}{3}x^3$ is an antiderivative of x^2 (check that $F'(x) = x^2$), $\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}$.

- $L_4 \le \frac{1}{3} \le R_4$, and L_4 is a slightly better approximation to the area, as expected since $f(x) = x^2$ is increasing and concave up on the interval.

COMMON INTEGRALS

$\int k\,dx = kx + C$	$\int 0\,dx = C$
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ if $n \ne -1$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \dfrac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \dfrac{1}{x^2+1}\,dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

TECHNIQUES OF INTEGRATION

Unlike differentiation, integration is "hard"—there are easy-to-write functions that don't have easy antiderivatives. The art of integration requires a bag of tricks. These are some of them. **NOTE:** All techniques work with both definite and indefinite integrals; pay special attention to the limits of integration. **TIP:** All functions that you will work with are integrable except at points where they blow up; all "smoothly" changing functions are differentiable.

SUBSTITUTION RULE

If $u = g(x)$ is continuously differentiable on some interval and $f(x)$ is integrable on the range of $g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

- This is the analog of the **Chain Rule** for integrals (see the *Calculus I* SparkChart). It is useful for composite functions and products.

Ex: $\int x^2\sqrt{x^3+8}\,dx$.

x^2 is a lot like $\frac{d(x^3+8)}{dx}$. Let $u = x^3 + 8$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{du}{3}$.

Substituting, we transform the original integral into $\int \frac{1}{3}\sqrt{u}\,du$, or

$\int \frac{1}{3}u^{\frac{1}{2}}\,du = \frac{2}{9}u^{\frac{3}{2}} + C = \frac{2}{9}\left(x^3+8\right)^{\frac{3}{2}} + C.$

- When evaluating definite integrals using substitution, you have a choice about how to deal with the limits of integration. Let's say that you're integrating with respect to x.
  1. *Less thinking:* Choose a useful u, substitute for x, integrate in terms of u, substitute x back, evaluate the integral with original limits.

     Ex: $\displaystyle\int_1^2 x^2\sqrt{x^3+8}\,dx = \frac{2}{9}\left(x^3+8\right)^{\frac{3}{2}}\Big]_1^2 = \frac{2}{9}16^{\frac{3}{2}} - \frac{2}{9}9^{\frac{3}{2}} = \frac{74}{9}.$

  2. *Less work:* Choose a useful $u = g(x)$ substitute, integrate in terms of u, then evaluate the integral using, for limits, values of $u = g(x)$ evaluated at the original limits of integration—all without substituting x back in. Formally, if $u = g(x)$, then

     $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} u\,du.$$

     In the example above, $\displaystyle\int_{x=1}^{x=2} x^2\sqrt{x^3+8}\,dx = \int_{u=g(1)}^{u=g(2)} \frac{1}{3}\sqrt{u}\,du = \frac{2}{9}u^{\frac{3}{2}}\Big]_9^{16} = \frac{74}{9}.$

INTEGRATION BY PARTS

Some function products (or quotients) cannot be integrated by substitution alone. Integration by parts works when one piece of the product has a simpler derivative and the other piece is easy to integrate.

- This is the integral analog of the **Product Rule** $\frac{d(fg)}{dx} = f'g + fg'$.
- **Indefinite integrals:** $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$, or $\int u\,dv = uv - \int v\,du$.
- **Definite integrals:** Slap on limits: $\int_a^b f(x)g'(x)\,dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$.
- A polynomial multiplied by a trigonometric, exponential, or logarithmic function is frequently best integrated by parts. Let the polynomial be u.

Ex: $\int (2x+1)e^{-x}\,dx$.

Let $u = 2x + 1$ (so $du = 2\,dx$), and $dv = e^{-x}dx$ (so $v = -e^{-x}$). The integral becomes

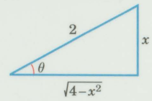
$$(2x+1)\left(-e^{-x}\right) - \int -2e^{-x}\,dx,$$

which simplifies to $-(2x+1)e^{-x} - 2e^{-x} + C$, or $(-2x-3)e^{-x} + C$.

TRIGONOMETRIC SUBSTITUTIONS

Square roots of quadratics, such as $\sqrt{a^2-x^2}$, cannot be integrated with the substitution $u = a^2 - x^2$ because the factor of du is missing. Enter trig substitutions, applying the substitution rule backwards. Trig substitutions are often necessary when calculating areas bounded by conic sections.

Ex: $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$. We can set $x = 2\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



Then $dx = 2\cos\theta\,d\theta$,

and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$

(since $\cos\theta \ge 0$ on the interval). The integral becomes

$\sin\theta = \frac{x}{2}$ $\quad\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$$\int \frac{2\cos\theta}{4\sin^2\theta}2\cos\theta\,d\theta = \int \cot^2\theta\,d\theta = \int \left(\csc^2\theta - 1\right)d\theta = -\cot\theta - \theta + C.$$

If we want to convert back from θ to x, we note that $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{4-x^2}}{x}$; thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

TABLE OF TRIGONOMETRIC SUBSTITUTIONS

Expression	Trig substitution	Expression becomes	Range of θ	Pythagorean identity used
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ $dx = a\cos\theta\,d\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $(-a \le x \le a)$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\,d\theta$	$\sqrt{x^2-a^2} = a\tan\theta$	$0 \le \theta < \frac{\pi}{2}$ (when $x > 0$) $\pi \le \theta < \frac{3\pi}{2}$ (when $x < 0$)	$\sec^2\theta - 1 = \tan^2\theta$
$\sqrt{x^2+a^2}$	$x = a\tan\theta$ $dx = a\sec^2\theta\,d\theta$	$\sqrt{a^2+x^2} = a\sec\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$

- *Pay careful attention to the limits of integration. The intervals for θ correspond to the ranges of the inverse trigonometric functions.*
- *Expressions of the form $\sqrt{\pm x^2 + bx + c}$ can be integrated by completing the square and converting to the form $\sqrt{\pm(x+h)^2 \pm a^2}$, where $h = \pm\frac{b}{2}$ and $a = \sqrt{\left|c - \frac{b^2}{4}\right|}$. Then choose the appropriate trig substitution depending on the $\pm$ signs.*

TECHNIQUES OF INTEGRATION (CONTINUED)

PARTIAL FRACTIONS

Integrating **rational functions**—ratios of polynomials—can be tricky. However, after factoring the denominator into linears and quadratics (which can always be done, though the coefficients may not be rational numbers), a rational function can be expressed as a sum of simpler **"partial" fractions.** These come in four "easy"-to-integrate types:

1. $\int \frac{du}{u} = \ln|u| + C.$ So $\int \frac{A\,dx}{ax+b} = \frac{A}{a}\ln|ax+b| + C.$

2. $\int \frac{du}{u^n} = \frac{u^{n+1}}{n+1} + C$ if $n \neq 1$. So $\int \frac{A\,dx}{(ax+b)^n} = \frac{A}{a(n+1)}(ax+b)^{n+1} + C.$

3. $\int \frac{du}{u} = \ln|u| + C.$ So

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \frac{A}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \int \frac{B-\frac{Ab}{2a}}{ax^2+bx+c}dx$$

$$= \frac{A}{2a}\ln|ax^2+bx+c| + \int \frac{D}{ax^2+bx+c}dx, \text{ where } D = B - \tfrac{Ab}{2a}.$$

4. $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C.$ So $\int \frac{D\,dx}{ax^2+bx+c}$ where the denominator has no real roots ($b^2-4ac<0$) can be evaluated by completing the square in the denominator, which becomes

$a(x+h)^2+k^2$ where $h = \frac{b}{2a}$ and $k = \sqrt{c - \frac{b^2}{4a}}$.

The step-by-step process for integrating $f(x) = \frac{p(x)}{q(x)}$ follows:

1. If necessary, **use long division** to get to the point where the degree of the numerator is less than the degree of the denominator. The function has the form $f(x) = s(x) + \frac{r(x)}{q(x)}$.
2. **Factor the denominator,** reducing it to linear factors in the form $(ax+b)^n$ and irreducible quadratic factors in the form $(cx^2+dx+e)^m$ where $d^2-4ce<0$.
3. **Decompose $\frac{r(x)}{q(x)}$ into a sum of partial fractions:**
   - If $q(x)$ has no repeated factors, express

     $$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_\ell}{a_\ell x+b_\ell} + \frac{C_1x+D_1}{c_1x^2+d_1x+e_1} + \cdots + \frac{C_kx+D_k}{c_kx^2+d_kx+e_k}.$$

     Solve for all the As, Cs, and Ds by multiplying the equation by $q(x)$ and equating coefficients.

     TIP: If $q(x)$ factors as a product of two linears $(x-a)(x-b)$, then we can solve for A and B in $\frac{r(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ quickly by taking $A = \frac{r(a)}{a-b}$ and $B = \frac{r(b)}{b-a}$.

   - If $q(x)$ has repeated factors, then for each factor in the form $(ax+b)^n$ expect fractions

     $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

     Each $(cx+dx+e)^m$ factor will give fractions $\frac{C_1x+D_1}{cx^2+dx+e} + \cdots + \frac{C_mx+D_m}{(cx^2+dx+e)^m}$.
4. **Integrate** $s(x)$ and each partial fraction individually, using the four types of integrals above.

Ex: $\frac{4x^3-17x^2-28}{(x-2)(x+2)(3x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{3x^2+4}$.

Cross-multiplying, simplifying, and equating coefficients gives the four equations $-2(A-B)+D=7$, $6(A-B)+D=-17$, $3(A+B)+C=4$, $(A+B)-C=0$. Solving this system of four linear equations (in this case, it is easier to view this as two systems of two equations, and solve for $A+B$ and C, and for $A-B$ and D independently), we get $A=-1$, $B=2$, and $Cx+D = x+1$. Now we can integrate.

POLYNOMIALS IN TRIGONOMETRIC FUNCTIONS

For powers of trigonometric functions, regular u-substitution may not work. Use the following substitutions instead.

- **Pythagorean identities:**

  $\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$
- **Square sine/cosine substitutions** (from the half-angle formulas):

  $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

- **Odd powers of sine or cosine:**

  To compute $\int \sin^n\theta\,d\theta$ when n is odd, keep one sine factor and replace the rest with cosines using $\sin^2\theta = 1-\cos^2\theta$ to obtain

  $$\int (1-\cos^2\theta)^{\frac{n-1}{2}}\sin\theta\,d\theta.$$

  Integrate using the substitution $u = \cos\theta$.
  For odd powers of cosine, "co" all the functions above ("co" cosine = sine).

- **Even powers of sine or cosine:** Use the square formulas

  $$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \text{ or } \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

  to reduce the power by half. Expand the expression (using the **Binomial Theorem**) and use appropriate tricks for odd or even powers on each factor individually.

- **Mixed powers of sine and cosine:** When computing $\int \sin^n\theta\cos^m\theta\,d\theta$, use combinations of substitutions outlined above. The end goal is always to reduce to a sum of terms with only one power of either $\sin\theta$ or $\cos\theta$ (or both) each, which can be integrated using a u-substitution.

- **Products of sines and cosines of different angles:** Use the following identities to get rid of products:

  $$\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$$

  $$\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$$

  $$\cos A\cos B = \frac{1}{2}(\cos(A-B)+\cos(A+B))$$

MNEMONIC: Products of like terms use cosines; unlike terms use sines.

- **Powers of secant or tangent:**
  - **Even powers of** $\sec\theta$: Convert all but two secants to tangents using Pythagorean identity $1+\tan^2\theta = \sec^2\theta$; use substitution $u=\tan\theta$.
  - **Odd powers of** $\tan\theta$: Convert all but one tangent to secants, pull out a factor of $\sec\theta$ from the polynomial in secants and use the substitution $u = \sec\theta$.

Ex: $\int \tan^5\theta\,d\theta = \int (1+\sec^2\theta)^2\tan\theta\,d\theta$

$$= \int\left(\frac{1}{\sec\theta} + 2\sec\theta + \sec^3\theta\right)\tan\theta\sec\theta\,d\theta = \int \frac{1}{u} + 2u + u^3\,du.$$

- **Other powers:** Combine tricks and use $\int \tan\theta\,d\theta = \ln|\sec\theta| + C$ and $\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$.

IMPROPER INTEGRALS

Improper integrals come in two types:

1. Definite integrals over an infinite interval. **Ex:** $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.
2. Definite integrals over an interval in which the function blows up to infinity (has a vertical asymptote). **Ex:** $\int_0^1 \frac{1}{x}\,dx$.

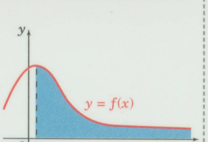
Not all improper integrals **converge**—represent a finite area. To evaluate an improper integral, we interpret it as a limit. If a finite limit exists, the integral converges; otherwise the integral **diverges**.

INTEGRALS OVER AN INFINITE INTERVAL

Improper integrals of this type should be rewritten as one of three limit forms:

1. Interval infinite to the right: $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$
2. Interval infinite to the left: $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$
3. Integrals over the whole real line:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \text{ for any } a.$$



Improper integral $\int_a^\infty f(x)\,dx$

The original integral converges only if *both* integrals over half-intervals converge.

- $\int_1^\infty \frac{dx}{x^r}$ converges (and equals $\frac{1}{r-1}$) if and only if $r>1$.
- The integral $\int_a^\infty f(x)\,dx$ will converge only if $\lim_{x\to\infty} f(x) = 0$ ($y=0$ is a horizontal asymptote). If the function does not tend to zero, the area underneath it will certainly not be finite, and the integral will diverge. Analogous statements are true for other infinite-interval improper integrals.

HOWEVER: $\lim_{x\to\infty} f(x) = 0$ alone does not imply convergence of an improper integral.

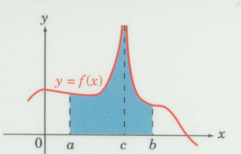
The classic example is $\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}\left(\ln|x|\Big]_1^t\right) = \lim_{t\to\infty}\ln t = \infty$.
The integral diverges.

INTEGRALS OVER AN INFINITE DISCONTINUITY

The integral $\int_a^b f(x)\,dx$ is improper if at any point c in the closed interval $[a,b]$, the function blows up.

- If $c=a$ is the left endpoint, then the integral is improper if $\lim_{x\to a^+} f(x) = \pm\infty$.

  The integral is interpreted as $\lim_{t\to a^+}\int_t^b f(x)\,dx$.



$f(x)$ has a vertical asymptote at $x=c$, so $\int_a^b f(x)\,dx$ is an improper integral.

- If $c=b$ is the right endpoint then the integral is improper if $\lim_{x\to b^-} f(x) = \pm\infty$

  and is reinterpreted as $\lim_{t\to b^-}\int_a^t f(x)\,dx$.
- If $c\in(a,b)$, then the original integral is understood to be the sum of the two improper integrals

  $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

  The original integral converges only if both endpoint-improper integrals converge independently.

- $\int_0^1 \frac{dx}{x^r}$ converges (and is equal to $\frac{1}{1-r}$) if and only if $r<1$.

NOTE: $\int_{-1}^1 \frac{dx}{x}$ diverges (and does not evaluate to 0) because both $\int_{-1}^0 \frac{dx}{x}$ and $\int_0^1 \frac{dx}{x}$ diverge, even though the two areas seem to be "equal" and opposite in sign.

On the other hand, $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = 0$ because both half-integrals $\int_{-1}^0 \frac{dx}{\sqrt[3]{x}}$ and $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ converge—and their values are equal in magnitude and opposite in sign.

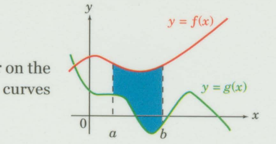
GEOMETRY OF CURVES

AREAS BOUNDED BY CURVES

Suppose that $f(x) \geq g(x)$ on the interval $[a, b]$ and both functions are continuous. Then the area bounded by the two curves $y = f(x)$, $y = g(x)$ and the two vertical lines $x = a$ and $x = b$ is

$$\int_a^b (f(x) - g(x))\, dx.$$

- In general, if the (continuous) curves cross each other on the interval, then the positive area defined by the curves between a and b is $$\int_a^b |f(x) - g(x)|\, dx.$$ It is most easily evalutated by considering subintervals whose endpoints are all points c such that $f(c) = g(c)$.



Shaded area is $\int_a^b (f(x) - g(x))\, dx$.

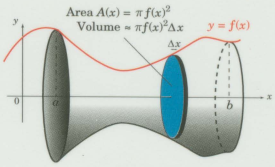
- If the area is bounded by horizontal lines, it may be easier to rewrite the curves in the form $x = f^{-1}(y)$ and integrate the difference between them with respect to y.

VOLUMES: SOLIDS OF REVOLUTION

Suppose that a solid is oriented along the x-axis so that the area of a **cross-section** (the slice of solid intersecting with a plane perpendicular to the x-axis) is given by the function $A(x)$. The volume of a slice of thickness Δx is $A(x)\Delta x$, and the volume of the solid bounded by the planes $x = a$ and $x = b$ is

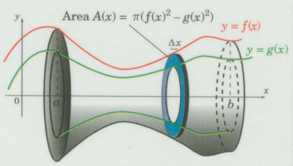
$$V = \int_a^b A(x)\, dx.$$

- **Disk method:** The volume of the solid swept out by the curve $y = f(x)$ as it revolves around the x-axis between $x = a$ and $x = b$ is given by $$\int \pi(\text{radius})^2\, dx \text{ or } \pi \int_a^b (f(x))^2\, dx.$$



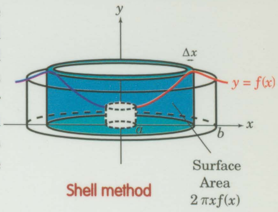
Disk method

- **Washer method:** If $f(x) > g(x)$ between a and b, then the volume of the solid swept out between the two curves $y = f(x)$ and $y = g(x)$ as they revolve around the x-axis between $x = a$ and $x = b$ is $$\int \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \text{ or}$$ $$\pi \int_a^b (f(x))^2 - (g(x))^2\, dx.$$



Washer method

- **Shell method:** A solid is obtained by revolving the region under the curve $y = f(x)$ between $x = a$ and $x = b$ (the area of this region is $\int_a^b f(x)dx$) around the y-axis. Instead of cross-sectional slabs perpendicular to the axis of revolution we consider the volume of a small cylindrical shell of radius x and thickness Δx. The surface area of a cylinder is (circumference) $\times$ (height) or $2\pi x f(x)$; thus, the volume of the solid is $$V = \int_a^b 2\pi x f(x)\, dx.$$



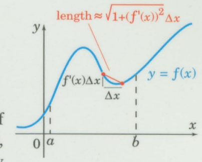
The shell method is often used when it is hard to compute the inside or outside radius of the cross-sectional slabs perpendicular to the axis of revolution.

ARC LENGTH

If $f(x)$ has a continuous derivative on the interval (a, b), then the length of the curve from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

In Leibniz notation this becomes $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$



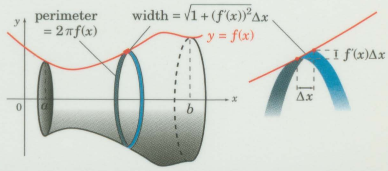
- Why? If we break up the interval into n subintervals each of width $\Delta x = \frac{b-a}{n}$ with x_k^* a sample point in the k^{th} interval, then the length of the curve on the k^{th} interval is approximately the length of the vector $(\Delta x, \Delta x f'(x_k^*))$, or $\Delta x\sqrt{1 + (f'(x_k^*))^2}$. Take the limit of the Riemann sum to get the formula.

SURFACE AREA: SOLIDS OF REVOLUTION

The surface area of a surface swept out by revolving the function $y = f(x)$ about the x-axis between $x = a$ and $x = b$ is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

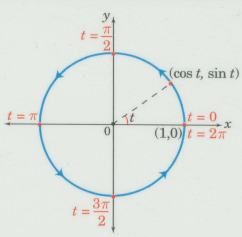
- The formula is obtained by approximating the surface area by cylindrical bands of radius $f(x)$ and width equal to the tiny arc length on the tiny interval.



PARAMETRIC CURVES

A **parametric curve** defines both the x- and the y-coordinates in terms of a third variable, often t (as in "time"). Parametric curves don't necessarily represent functions and don't have to pass the vertical line test. Sometimes the domain of t is restricted to an interval $a \leq t \leq b$.

Ex: $x = \cos t$ and $y = \sin t$ for $0 \leq t < 2\pi$ are **parametric equations** that describe the unit circle.



- To convert a curve described parametrically to Cartesian coordinates, try to relate x and y directly, eliminating t. If possible, solve for t in one of the equations, and plug that expression into the other equation. Again, this may not give a function for y in terms of x.

  Ex: In the example above, solving and plugging in will give something like $y = \sin(\cos^{-1} x)$, which is not very useful. However, if you use the fact that $\cos^2\theta + \sin^2\theta = 1$ for all angles θ, you can relate x and y with the familiar $x^2 + y^2 = 1$ equation for the unit circle. To define a function, you have to choose a piece of the curve; for example, $y = \sqrt{1 - x^2}$. Doing so is equivalent to restricting t to the interval $[0, \pi]$.

GEOMETRY OF THE CURVE

The following formulas are obtained using the chain rule.

- **Slope of tangent:** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$ If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$, then the tangent is vertical.
- **Concavity:** $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$
- **Area defined by curve:** The area between the x-axis and the curve traced out from $t = a$ to $t = b$ is $$A = \int_a^b y(t)x'(t)dt.$$

NOTE: The area is counted as negative for the regions where the curve is moving "backwards"—i.e., $\frac{dx}{dt} < 0$—and positive when the curve is moving "forwards."

- *HAPPY CONSEQUENCE:* The area enclosed by a loop wholly above the x-axis, traversed exactly once from $t = a$ to $t = b$, can be computed directly: $$A = \left|\int_a^b y(t)x'(t)dt\right|.$$ The integral is positive for loops traced out clockwise, negative for those traced counterclockwise. Alternatively, break up $[a, b]$ into subintervals depending on the sign of $\frac{dx}{dt}$; integrate separately.

- **Arc length:** The length of a parametric curve traced out from $t = a$ to $t = b$ is $$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$ This formula also works works for loops.
- **Surface area of revolved solid:** If the same curve always stays above the x-axis ($y(t) \geq 0$), then the surface area swept out when it is revolved around the x-axis is $$S = \int_a^b 2\pi y(t)\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

POLAR COORDINATES

For more on polar coordinates, see the Pre-calculus SparkChart.

Polar coordinates describe a point $P = (r, \theta)$ on a plane in terms of its distance r from the **pole** (usually, the origin O) and the (counterclockwise) angle θ that the line $\overline{OP}$ makes with a reference line (usually, the positive x-axis).

- To identify a point, it is standard to limit $r \geq 0$ and $0 \leq \theta < 2\pi$, although $(-r, \theta) = (r, \theta \pm \pi)$ and $(r, \theta) = (r, \theta + 2n\pi)$ for integer n.

CONVERTING BETWEEN CARTESIAN AND POLAR COORDINATES

- **From Cartesian to polar:** $r = \sqrt{x^2 + y^2}$; $\theta = \tan^{-1}\frac{y}{x}$
- **From polar to Cartesian:** $x = r\cos\theta$; $y = r\sin\theta$

FUNCTIONS

Functions in polar coordinates usually define r in terms of θ. They need not (and almost never will) pass the vertical line test.

Circles: The graph of $r = a$ is a circle of radius $|a|$ centered at the origin. The graphs of $r = a\sin\theta$ and $r = a\cos\theta$ are circles of radius $\left|\frac{a}{2}\right|$ centered at $(0, \frac{a}{2})$ and $(\frac{a}{2}, 0)$, respectively.

Roses: The graphs of $r = \sin n\theta$ and $r = \cos n\theta$ are roses centered at the origin with n petals if n is odd, $2n$ petals if n is even.

Limaçons and cardiods: The graphs of $a \pm b\sin\theta$ and $a \pm b\cos\theta$ are **limaçons**. If $\left|\frac{b}{a}\right| > 1$, the limaçon has an inner loop; if $\left|\frac{b}{a}\right| = 1$, the limaçon is "heart-shaped" and is called a **cardiod**.

GEOMETRY OF THE CURVE

- **Slope of tangent:** Convert the curve $r = f(\theta)$ into parametric equations in Cartesian coordinates: $x = r\cos\theta = f(\theta)\cos\theta$; $y = f(\theta)\sin\theta$. The slope of the tangent to the curve at $(x(\theta), y(\theta))$ is $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}.$$
- **Area:** The area enclosed by rays at $\theta = \alpha$ and $\theta = \beta$ bounded by the curve $r = f(\theta)$ is $$A = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta.$$

Why? The area of a circle of radius r is πr^2 (angle sweep of 2π). The area of a sector of a circle of radius r and angle measure θ is thus $\frac{1}{2}r^2\theta$. We approximate the area by a Riemann sum of sliver-sectors with radius r and angle measure $\Delta\theta$.

- **Arc length:** The length of an arc $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is $$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$ The formula is derived by converting the curve to Cartesian parametric equations with θ as the parameter.

INTEGRATION: APPLICATIONS TO PHYSICS

WORK

Newton's Second Law, $F = ma$, states that the force F on an object is proportional to the object's mass m and its acceleration $a = \frac{d^2x}{dt^2}$.

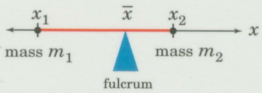
Work is the product of a force and the distance through which it acts. If the force is constant, then $W = Fx$. If the force $F(x)$ is variable and depends on the distance x, then the work done by $F(x)$ in moving an object from $x = a$ to $x = b$ is $$W = \int_a^b F(x)\,dx.$$

- The classic situation of a force dependent on distance is the subject of **Hooke's Law:** the force required to stretch or compress a spring x units away from its natural position is given by $F(x) = kx$; here, k is a constant that depends on the tightness of the spring.

CENTER OF MASS

The **center of mass** (CM) of any system is the point on which (if connected) it could balance on a **fulcrum**.

- **Moment:** The farther away something is from the fulcrum, the "heavier" its mass counts. Each mass of weight m a distance x from some point contributes mx worth of **moment** (or **torque**) *with respect to that point*. If the point is the CM, then all the moments of the system have to balance.
  The system behaves as though all of its mass were concentrated at the CM.



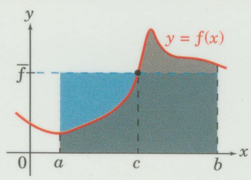
Torques: $m_1(\overline{x} - x_1)$ counterclockwise
$m_2(x_2 - \overline{x})$ clockwise
Masses balance, so $\overline{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$.

- **2 masses, 1 axis:** If a massless rod with objects of masses m_1 and m_2 at each end balances on a fulcrum at its CM, then the distances d_1 and d_2 from the masses to the fulcrum must satisfy $m_1d_1 = m_2d_2$.
  - If such a rod has length d, then distance d_1 from the m_1 mass to the fulcrum satisfies $m_1d_1 = m_2(d - d_1)$. Solving, $d_1 = \frac{m_2d}{m_1+m_2}$. We can view this as positioning the rod along the x-axis with the m_1 mass at the origin. The CM, then, is at the point $\frac{m_1x_1+m_2x_2}{m_1+m_2}$, where x_1 and x_2 are the distances of the masses from the origin. Here, $x_1 = 0$ and $x_2 = d$, the length of the rod.
- **Discrete masses, 1 axis:** In general, a system of n objects of masses $m_1, m_2, \ldots, m_n$ positioned at points $x_1, x_2, \ldots, x_n$ along the x-axis (respectively) has CM at the point $$\overline{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}.$$
  This is the **moment of the system about the *x*-axis.**
  The **moment of the system about the *y*-axis** can be computed independently.
- **Discrete masses, 2 axes:** The CM of a system of objects of masses $m_1, \ldots, m_n$ at points $(x_1, y_1), \ldots, (x_n, y_n)$ on a coordinate system is at the point $$(\overline{x}, \overline{y}) = \left(\frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots m_n}, \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots m_n}\right).$$
- **Continuous mass, uniform density:** The CM of a flat plate-like object of uniform density ρ is computed by taking the limit of a Riemann sum. If its area is A, the total mass is given by $m = \rho A$. Suppose that the shape of the object is given by the curve $y = f(x)$ from $x = a$ to $x = b$. As usual, we approximate the object by thin rectangular strips of width Δx, height $f(x)$, area $f(x)\Delta x$, mass $\rho f(x)\Delta x$, and CM at the point $\left(x, \frac{1}{2}f(x)\right)$.
  - ***x*-coordinate of the CM:** For each strip, the moment is given by $$(\text{mass}) \times (x\text{-coordinate of strip CM}) = \rho x f(x)\Delta x.$$
    The x-coordinate of the CM (equivalently, the moment about the y-axis) is therefore $$\frac{\rho}{m}\int_a^b x f(x)\,dx = \frac{1}{A}\int_a^b x f(x)\,dx.$$
  - ***y*-coordinate of the CM:** The moment of each strip is $$(\text{mass}) \times (y\text{-coordinate of strip CM}) = (\rho f(x)\Delta x)\left(\tfrac{1}{2}f(x)\right).$$
    The y-coordinate of the CM is therefore $\frac{1}{A}\int_a^b (f(x))^2\,dx.$
    The density ρ doesn't appear in the final result; all that matters is that the density is uniform.

INTEGRATION: APPLICATIONS TO PROBABILITY & STATISTICS

AVERAGE VALUE

For a discrete set of values, their average multiplied by their number gives their sum. The analog of an average for a continuous function $f(x)$ on the interval $[a, b]$ is the **average value** $\overline{f}$, which has the property that the rectangle of height $\overline{f}$ and width $b - a$ has the same area as is enclosed under the curve $y = f(x)$. Thus $$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$



$\overline{f} = f(c)$ is the average value of $f(x)$ on the interval $[a, b]$. The two shaded regions have equal area.

- **The Mean Value Theorem for Integrals** states that a continuous function attains its average value. Like the MVT for derivatives (see the *Calculus I* SparkChart), this is a completely intuitive statement.

GENERAL PROBABILITY DENSITY

A **probability density function** describes how likely it is that the outcome of some "trial" is x. The probability that the outcome is any specific point a is negligible; instead, we talk about the chances that the outcome falls in some range and think of areas under the probability density curve as representing actual probabilities. The probability that the outcome is between a and b is $$\int_a^b f(x)\,dx.$$

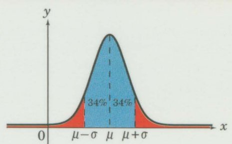
- The probability that the outcome is *something* is 1; therefore, $\int_{-\infty}^{\infty} f(x)\,dx = 1.$
- The **mean** of a probability density function is the long-run average outcome; it can be seen as the x-coordinate of the CM of the region on a graph and is given by $\mu = \int_{-\infty}^{\infty} x f(x)\,dx.$
- The **median** is the point m such that the probability that $x < m$ is equal to the probability that $x > m$. (Again, the probability that $x = m$ is negligible.) Solve for m in the equation $\int_m^\infty f(x)\,dx = \frac{1}{2}$ or $\int_{-\infty}^m f(x)\,dx = \frac{1}{2}$.

THE NORMAL DISTRIBUTION

The normal distribution, or "bell curve," is a probability density that often arises from repeated random events.

The probability density $N(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

- The **mean** is μ.
- The variable σ is the **standard deviation,** a measure of how clustered the outcomes are around the mean. The probability that the outcome is within σ of the mean is about 68% : $$\int_{\mu-\sigma}^{\mu+\sigma} N(x)\,dx \approx 0.68.$$
  The probability that the outcome is within $2\sigma$of the mean is about 95%.



Normal distribution with mean μ and standard deviation σ. The blue region is 68% of the total shaded area.

DIFFERENTIAL EQUATIONS

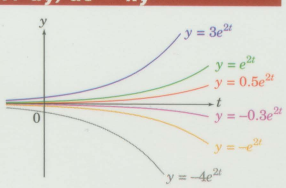
An (ordinary) **differential equation** (diff-eq) involves the derivative(s) of a (single-variable) function.

- The **order** of a diff-eq is the highest degree of a derivative involved in the equation. $y^3 = y'' + y' + x$ is a second-order diff-eq.
- A **solution** to a diff-eq is any curve $y = f(x)$ which satisfies the diff-eq. A **general solution** is the complete family of curves that satisfy the diff-eq. **Ex:** The general solution to the diff-eq $y' = 4\sin 2x$ is $y = -2\cos 2x + C$.
- An **initial condition**, often the value of $y(0)$, isolates a **particular solution** from the family of general solutions. **Ex:** If $y' = 4\sin 2x$ and $y(0) = 3$, then $y = -2\cos 2x + 5$.

EXPONENTIAL GROWTH AND DECAY: *dy/dt = ky*

$\frac{dy}{dt} = ky$ is a common type of diff-eq. The general solution is $y = Ae^{kt}$.

- **Solution:** Separating and rewriting, we get $\int \frac{dy}{y} = \int k\,dt$. Integrating yields $\ln|y| = kt + C$ or $\pm y = e^{kt+C}$. Since e^C is a positive multiplicative factor, we replace $\pm e^C$ by the constant A and rewrite $y = Ae^{kt}$.
  - If $k > 0$, the solution represents exponential growth; if $k < 0$, exponential decay.
  - A is the initial value of the function at $t = 0$.



Several solutions to the differential equation $\frac{dy}{dt} = 2y$

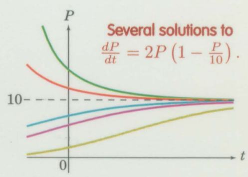
Word problems that often reduce to diff-eqs of this type:

- **Unlimited population growth:** k is called the **relative growth rate**; 1% growth (per year) means $k = 0.01$ (if t is measured in years).
- **Radioactive decay:** The function measures the mass remaining at time t. The constant k is negative; it is often conveyed in terms of the (constant) **half-life** of the element—the amount of time it takes for half of the remaining mass of the element to decay. If h is the half-life, then $k = -\frac{\ln 2}{h}$.
- **Compounded interest:** The final value P of an investment compounded n times a year with initial value P_0 and yearly interest r after t years is $P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously $(n \to \infty)$, then value is $\lim_{n\to\infty} P(t) = P_0e^{rt}$. The (continuously compounded) investment is changing at a rate proportional to its value.

SEPARABLE DIFFERENTIAL EQUATIONS

A diff-eq is called **separable** if it is a first-order equation that can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$, where f and g depend only on one variable. The exponential growth diff-eq $y' = ky$ is separable.

- To solve a separable diff-eq, we abuse Leibniz notation to rewrite it as $\frac{dy}{g(y)} = f(x)\,dx$ and integrate each side separately. Only one constant C is necessary.
- **Ex: Mixing problems:** A tank filled with a solution of one concentration is draining at one rate while a solution of different concentration is being pumped in at another rate. The rate of change of the concentration y at time t is given by $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$. The rate out is proportional to the current concentration.
- **Ex: Logistic (limited) population growth:** More accurately represents population growth taking into account limited natural resources. A population $P(t)$ with natural growth rate k and maxiumum **carrying capacity** $P_{\max}$ will satisfy the logistic differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{P_{\max}}\right)$.
  - To solve: Rewrite as $k\,dt = \frac{P_{\max}}{P_{\max}P - P^2}dP$ and integrate using partial fractions to obtain the general solution $P(t) = \frac{P_{\max}}{1 - Ae^{-kt}}$ where $A = \frac{P(0) - P_{\max}}{P(0)}$ is the initial condition.



Several solutions to $\frac{dP}{dt} = 2P\left(1 - \frac{P}{10}\right)$.

- If $P(0) = 0$ or $P(0) = P_{\max}$, then the original diff-eq gives $\frac{dP}{dt} = 0$; the population is stable.
- If $P(0) \neq 0$, then $\lim_{t\to\infty} P(t) = P_{\max}$, which makes sense.

LINEAR EQUATIONS

A **linear differential equation** is an inseparable equation of the form $y' + f(x)y = g(x)$, with $f(x), g(x)$ continuous functions.

- To solve, multiply both sides by the "integrating factor" $u(x) = e^{\int f(x)dx}$; note that $\frac{du}{dx} = e^{\int f(x)dx}f(x) = u(x)f(x)$. Moreover, $\frac{d(uy)}{dx} = uy' + u'y = uy' + ufy$. This gives $\frac{d(uy)}{dx} = ug$; now solve for the function uy.
- The general solution is $y = \frac{1}{u(x)}\left(\int u(x)g(x)\,dx + C\right)$. (Note that the exponent in $u(x)$ can be *any* of the family of functions $\int f(x)dx$; the constant will drop out.)

SEQUENCES AND SERIES

Why is this a Calculus topic? Complicated functions can often be approximated with polynomials—or with infinite polynomials called "power series." Polynomials, even infinite ones, are easy to differentiate and integrate. So we can find an approximate integral or derivative of a complicated function by representing it as a power series.

SEQUENCES

A **sequence** is an ordered list of real numbers, called **terms.** An **infinite** sequence has infinitely many terms.

- Shorthand: $\{a_k\}_{k=1}^{\infty}$ represents the sequence $a_1, a_2, a_3, \ldots$.
- A sequence is defined **explicitly** if each of its terms can be found independently of the other terms. **Ex:** $a_n = n^2$ is the sequence $1, 4, 9, 16, \ldots$. A sequence is defined **recursively** if the n^{th} term is found using the preceding term(s). **Ex:** $a_1 = 1;\ a_n = a_{n-1} + (2n-1)$ is again the sequence $1, 4, 9, \ldots$.

Limit of a sequence:

- The **limit** of an infinite sequence, denoted $\lim\limits_{n\to\infty} a_n$, if it exists, is the value that the sequence approaches. If the limit exists and is finite, then the sequence is called **convergent.** If not, the sequence is **divergent.**
- Formally, the limit exists and is equal to a if for all $\varepsilon > 0$ there exists an N so that whenever $n > N$, we have $|a_n - a| < \varepsilon$.

  For a divergent sequence whose terms tend toward infinity, we can say that $\lim\limits_{n\to\infty} a_n = \infty$ if for all integers A there exists an N so that if $n > N$, then $a_n > A$.

- A sequence $\{a_n\}$ is called **increasing** if $a_k \le a_{k+1}$ for all k and **decreasing** if $a_k \ge a_{k+1}$ for all k. A sequence is called **monotonic** if it is either increasing or decreasing.
- A sequence is said to be **bounded above** if every term is smaller than some fixed constant and **bounded below** if every term is greater than some fixed constant. A sequence bounded both above and below is called simply **bounded**.
- **Monotonic Sequence Theorem:** All bounded, monotonic sequences are convergent. A bounded increasing sequence cannot increase too much; the terms must cluster around some limit.

SERIES: DEFINITIONS AND BASIC TYPES

A **series** is a summed sequence: $a_1 + a_2 + a_3 + \cdots$. An **infinite series** has infinitely many terms. An infinite series is often denoted $\sum\limits_{k=1}^{\infty} a_k$ or just $\sum a_k$.

- A **partial sum** of a series is a cut-off series sum $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$.
- The **sum** of a series exists if the sequence of partial sums converges to a limit sum. If the limit of partial sums exists, the series is called **convergent**, otherwise it is **divergent.**

A **geometric series** has the form $a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k$, where $a \ne 0$.

- It is convergent if and only if $|r| < 1$, in which case its sum is $\frac{a}{1-r}$.
- We can compute the partial sum $s_n = \sum\limits_{k=0}^{n} ar^k = \dfrac{a\left(1-r^{n+1}\right)}{1-r}$ if $r \ne 1$.

- A ***p*-series** has the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. It converges if and only if $p > 1$.

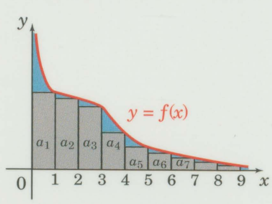
  The special divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **harmonic series.**

GENERAL TESTS FOR CONVERGENCE

- **Divergence test:** If $\lim_{n\to\infty} a_n \ne 0$ (or if the limit does not exist), then the series $a_1 + a_2 + a_3 + \cdots$ diverges.

Comparison tests: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- **Convergence:** If $\sum b_n$ converges, and $a_n \le b_n$ for all n, then $\sum a_n$ converges.
- **Divergence:** If $\sum b_n$ diverges and $a_n \ge b_n$ for all n, then $\sum a_n$ diverges.
- **Limit comparison test:** If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is positive, then either both series converge or both diverge.



Integral test: The sum of the infinite series (gray region) is strictly smaller than the area under $f(x)$ (blue and gray region). If $\int_1^{\infty} f(x)dx$ converges, then so does the series.

- **Integral test:** If $\{a_n\}$ is a monotonically decreasing positive sequence and $f(x)$ is a continuous function with the property that $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

ABSOLUTE CONVERGENCE AND ALTERNATING SERIES

A series $\sum a_n$ **converges absolutely** if the series of absolute values $\sum |a_n|$ converges. If the series of absolute values does not converge, but the original series does, then it **converges conditionally.**

- **Absolute convergence test:** If a series conveges absolutely, then it is convergent.
- **Ratio test:** Suppose $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = L$ exists and is finite.

  If $L < 1$, then the series converges absolutely. If $L > 1$ (or if the limit is infinite) then the series diverges. If $L = 1$ then the test is inconclusive.
- **Root test:** Suppose $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists and is finite.

  If $L < 1$, then the series converges absolutely. If $L > 1$ (or if the limit is infinite) then the series diverges. If $L = 1$ then the test is inconclusive.

TIP: If the ratio test is inconclusive on a particular series, then so is the root test. Try something else.

- **Alternating series test**: An **alternating series** has terms with alternating $\pm$ signs.

  If a_n are all positive, then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ will always converge if both
  1. $a_{n+1} \le a_n$ for all n, and
  2. $\lim_{n\to\infty} a_n = 0$.

These conditions are sufficient but not necessary. For an alternating series that satisfies these conditions, the error from truncation is always smaller than the next term.

GENERAL POWER SERIES

A **power series** is a formal function in the form of an infinite polynomial:

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Here, x is the variable and the a_n are coefficients; this series is "centered at a."

Many complex functions can be represented as power series; we need to know when these series converge. A power series about a can converge in one of three ways:

1. Only at $x = a$;
2. For all real x;
3. In an interval of radius R around a (i.e., $a - R < x < a + R$). R is called the **radius of convergence.** *NOTE:* The endpoints $a - R$ and $a + R$ have to be tested for each function.

When functions are represented as power series, they can be **integrated** or **differentiated** term by term in the usual way: if $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, then

- $f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, and
- $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1} + C$.

  The radii of convergence of $f'(x)$ and $\int f(x)dx$ are the same as that of $f(x)$ (but check endpoints individually).

Ex: The classic example is the power series for $\tan^{-1} x$:

- Start with the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ with radius of convergence 1.
- Substituting $x \to -x^2$, we get the power series for $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$, also with radius of convergence 1.
- Integrating, we get $\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.
- Finally, since $\tan^{-1} 0 = 0$, we know that $C = 0$. Check convergence at $x = \pm 1$ to find that the power series converges when $|x| \le 1$.

TAYLOR AND MACLAURIN SERIES

If $f(x)$ can be represented by a power series around a, then the coefficients a_n are given by

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Here, $f^{(n)}$ is the n^{th} derivative and $n! = 1 \cdot 2 \cdot 3 \cdots n$ with $0! \overset{\text{def}}{=} 1$.

- The **Taylor series for** $f(x)$ centered at a has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

  It converges at a or in some interval around a.
- The **Maclaurin series** for $f(x)$ is the Taylor series centered at 0, so $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$.

- **Arithmetic with Taylor series:** Functions written in Taylor series form can be added, subtracted, multiplied (painstakingly collecting like terms), and even divided if the constant term of the denominator is non-zero.
- **Polynomial approximations** to $f(x)$: The Taylor series for $f(x)$ about a can be used to approximate $f(x)$ by a polynomial of any degree for x-values near a. Ignore the higher-order terms. The linear polynomial is the tangent line to the curve at $x = a$.
- **Applications to limits:**

  Ex: $\lim\limits_{x\to 0} \dfrac{x - \tan^{-1} x}{x^3} = \lim\limits_{x\to 0} \dfrac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)}{x^3} = \lim\limits_{x\to 0} \left(\dfrac{1}{3} - \dfrac{x^2}{5} + \cdots\right) = \dfrac{1}{3}.$

- **Error bound:** Rule of thumb: The error of a truncated Taylor series is less than something a lot like the next term after the cut-off.
  - Formally, if the Maclaurin series for $f(x)$ converges at h, and $\left|f^{(n+1)}(x)\right| \le M$ for all $-h \le x \le h$, then the error in evaluating $f(h)$ by the Maclaurin series truncated after the n^{th} degree term is less than $\frac{Mh^{n+1}}{(n+1)!}$.

IMPORTANT MACLAURIN SERIES

Function	Series	Domain of convergence
$\dfrac{1}{1-x}$	$\sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$\lvert x\rvert < 1$
$\dfrac{1}{(1-x)^2}$	$\sum\limits_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + \cdots$	$\lvert x\rvert < 1$
$\ln(1-x)$	$-\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n} = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \cdots$	$-1 \le x < 1$
e^x	$\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	all real x
$\sin x$	$\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	all real x
$\cos x$	$\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	all real x
$\tan^{-1} x$	$\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$\lvert x\rvert \le 1$

BINOMIAL SERIES

The **binomial series** is the Maclaurin series for functions in the form $(1+x)^r$. It is finite for positve integers r, but works for all real numbers.

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2}x^2 + \cdots.$$

- Notation: $\binom{r}{n}$ = "r choose n" $= \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$.
  - Defined for all real r and non-negative integer n.
  - If r is an integer and $r < n$, then $\binom{r}{n} = 0$.
  - If r is a non-negative integer, then $\binom{r}{n}$ is the number of ways that a group of n objects can be chosen from a set of r objects.
  - If r is negative, then $\binom{-r}{n} = (-1)^n \binom{r+n-1}{n}$.

The infinite binomial series converge for $|x| < 1$. Convergence at ± 1 depends on r: if $r \ge 0$, then the series converges at ± 1; if $-1 < r < 0$, only at $x = 1$; otherwise at neither endpoint.